Note

# Limit(极限)

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**Definition:** Let $f(x)$ be a function defined on an open interval containing $c$ (except possibly at $c$) and let $L$ be a real number. The statement

$$\lim_{x\to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

if $\quad 0 < |x - c| < \delta$

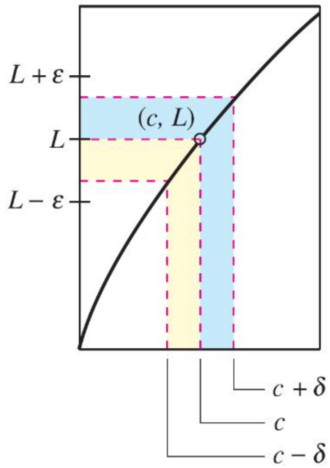
then $\quad |f(x) - L| < \varepsilon$

**定义：** 在$x_0$附近的开区间上定义的函数 $f(x)$,若存在常数 $L$ 使得，对于任意给定的$\varepsilon > 0$，总存在$\delta > 0$ 使得当 $0 < |x - x_0| < \delta$时，总有$|f(x) - L| < \varepsilon$，则 $L$ 称为函数 $f(x)$当$x \to x_0$时的极限，记为 $\lim\limits_{x\to x_0} f(x) = L$.

**简记：** $\lim\limits_{x\to x_0} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0, when\ 0 < |x - x_0| < \delta, |f(x) - L| < \varepsilon$

**Note 1: Arbitrarily close**

Arbitrarily close to $L$ means that for any small real number $\varepsilon$, $f(x)$ lies in the interval $(L - \varepsilon, L + \varepsilon)$ or $|f(x) - L| < \varepsilon$




**Note 2: x approaches c**

(for the arbitrarily $\varepsilon$ given above), there exists a positive number $\delta$ such that $x$ lies in either $(c - \delta, c)$ or $(c, c + \delta)$, or $0 < |x - c| < \delta$

**Note 3: How to read**

The limit of $f(x)$, as $x$ approaches $c$, equals to $L$.

**Note 4: *L* is a fixed real number**

a) $\lim\limits_{x\to 1}(x+1)(x-2) = -2$: -2 is a fixed number

b) $\lim\limits_{x\to 1}\frac{(x+1)}{(x-1)}$: $+\infty/-\infty$ are NOT fixed number, $\longrightarrow$limit not exist

**Note 5: The expression** $\lim\limits_{x\to c} f(x) = L$

implies

a) The limit exists

b) The limit is $L$

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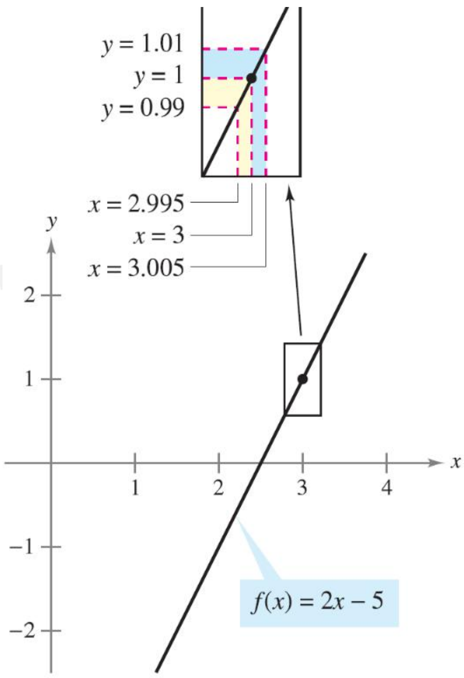



**Example 6:** Given the limit $\lim\limits_{x\to 3}(2x - 5) = 1$, find $\delta$ such that $|2x - 5| < 0.01$ whenever $0 < |x - 3| < \delta$

目标：求出$|x - c|$ $(c = 3)$的不等式

*Solution* (a) $f(x) = 2x - 5$, (b) $L = 1$, (c) $\varepsilon = 0.01$, (d) $c = 3$, find $\delta$.

Note

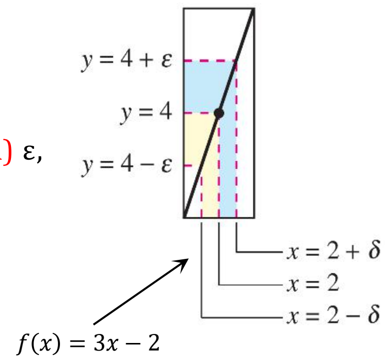
$|f(x)-L|<\varepsilon$

$\Leftarrow |(2x-5)-1|<0.01$

$\Leftarrow |2x-6| = 2|x-3|<0.01$

$\Leftarrow |x-3|<\dfrac{0.01}{2}=0.005=\delta$

**Example 7:** Use the the $\varepsilon-\delta$ definition of limit to prove that $\lim\limits_{x\to 2}(3x-2)=4$




目标：找出$|f(x)-L|$与$|x-c|$之间的关系，
即$|(3x-2)-4|$与$|x-2|$之间的关系。

*Solution* (a) $f(x)=3x-2$, (b) $L=4$, (c) c $=2$ (d) ε, find $\delta$.

For each $\varepsilon>0$, to prove that $|f(x)-L|<\varepsilon$

$\Leftarrow |(3x-2)-4|<\varepsilon$

$\Leftarrow |3x-6|<\varepsilon$

$\Leftarrow 3|x-2|<\varepsilon$

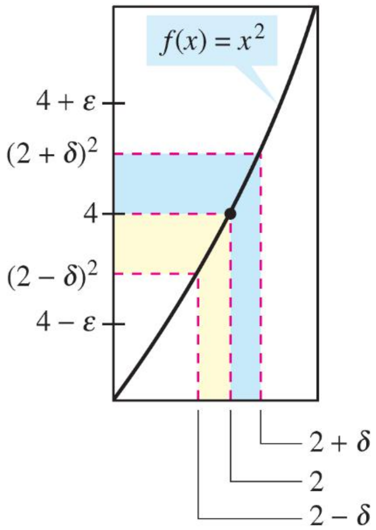
$\Leftarrow |x-2|<\varepsilon/3$

There exists a $\delta=\varepsilon/3$, so that $0<|x-2|<\delta=\varepsilon/3$

$\therefore \lim\limits_{x\to 2}(3x-2)=4$

倒 推 法：找$|x-c|$的表达式

**Example 8:** Use the the $\varepsilon-\delta$ definition of limit to prove that $\lim\limits_{x\to 1}(2x^2-1)=1$




*Solution* (a) $f(x)=2x^2-1$, (b) $L=1$, (c) c $=1$ (d) ε, find $\delta$.

For each $\varepsilon>0$, to prove that $|f(x)-L|<\varepsilon$

$\Leftarrow |(2x^2-1)-1|<\varepsilon$

$\Leftarrow |2x^2-2|<\varepsilon$

$\Leftarrow 2|x^2-1|<\varepsilon$

$\Leftarrow |x^2-1|<\varepsilon/2$

$\Leftarrow |x+1||x-1|<\varepsilon/2$

$$\Leftarrow \begin{cases} when\ x\ apporachs\ 1, |x+1|\approx 2<3 \\ \dfrac{\varepsilon}{2}=3\left(\dfrac{\varepsilon}{6}\right)=3\delta \end{cases}$$

There exists a $\delta=\varepsilon/6$, when $0<|x-1|<\delta=\varepsilon/6$, we have $0<|x+1||x-1|<\varepsilon/2$

$\therefore \lim\limits_{x\to 1}(2x^2-1)=1$

$|x-c|$

Example 8 的简化写法：

$\forall\varepsilon>0$, to prove $|f(x)-L|=|(2x^2-1)-1|<\varepsilon$

$\Leftarrow |(2x^2-1)-1|<\varepsilon \Leftarrow |2x^2-2|<\varepsilon \Leftarrow 2|x^2-1|<\varepsilon$

$\Leftarrow |x^2-1|<\varepsilon/2 \Leftarrow |x+1||x-1|<\varepsilon/2=3(\varepsilon/6)=3\delta$

$\exists\delta=\varepsilon/6>0$, s.t. $|x+1||x-1|<3(\varepsilon/6)=\varepsilon/2$

$\therefore \lim\limits_{x\to 1}(2x^2-1)=1$

**Example 9:** Quotient Function $\lim\limits_{x\to 1}\frac{3}{x-2}=-3$

*Solution* (a) $f(x)=\frac{3}{x-2}$, (b) $L=-3$, (c) c $=1$ (d) ε, find $\delta$.

For each $\varepsilon > 0$, to prove that $|f(x) - L| < \varepsilon$

$$\Leftarrow \left|\frac{3}{x-2} - (-3)\right| = 3\left|\frac{1}{x-2} + 1\right| < \varepsilon$$

$$\Leftarrow 3\left|\frac{1+(x-2)}{x-2}\right| < \varepsilon \Leftarrow \left|\frac{1+(x-2)}{x-2}\right| = \left|\frac{x-1}{x-2}\right| < \frac{\varepsilon}{3}$$

$|x - c|$

$$\Leftarrow \begin{cases} when\ x\ apporachs\ 1, \dfrac{1}{|x-2|} \approx 1 < 2 \\ \dfrac{\varepsilon}{3} = 2\left(\dfrac{\varepsilon}{6}\right) = 2\delta \end{cases}$$

There exists a $\delta = \frac{\varepsilon}{6}$ , when $0 < |x-1| < \delta = \frac{\varepsilon}{6}$, we have $0 < \left|\frac{x-1}{x-2}\right| < \frac{\varepsilon}{3}$

$$\therefore \lim_{x\to 1}(2x^2 - 1) = 1$$

**Example 10:** Radical function $\lim_{x\to 1}\sqrt{2x+3} = \sqrt{5}$

*<u>Solution</u>* (a) $f(x) = \sqrt{2x+3}$, (b) $L = \sqrt{5}$, (c) $c = 1$ (d) ε, find $\delta$.

For each $\varepsilon > 0$, to prove that $|f(x) - L| = |\sqrt{2x+3} - \sqrt{5}| < \varepsilon$

$$\Leftarrow \left|\frac{(\sqrt{2x+3}-\sqrt{5})(\sqrt{2x+3}+\sqrt{5})}{(\sqrt{2x+3}+\sqrt{5})}\right| < \varepsilon$$

$$\Leftarrow \left|\frac{(2x+3)-5}{(\sqrt{2x+3}+\sqrt{5})}\right| = \left|\frac{2x-2}{(\sqrt{2x+3}+\sqrt{5})}\right| = 2\left|\frac{x-1}{(\sqrt{2x+3}+\sqrt{5})}\right| < \varepsilon$$

$$\Leftarrow \left|\frac{x-1}{(\sqrt{2x+3}+\sqrt{5})}\right| < \frac{\varepsilon}{2}$$

$$\Leftarrow \begin{cases} when\ x \to 1, \dfrac{1}{\sqrt{2x+3}+\sqrt{5}} = \dfrac{1}{2\sqrt{5}} \approx 0.2 < 1 \\ \dfrac{\varepsilon}{2} = 1\left(\dfrac{\varepsilon}{2}\right) = 1\delta \end{cases}$$

There exists a $\delta = \frac{\varepsilon}{2}$ , when $0 < |x-1| < \delta = \frac{\varepsilon}{2}$ , we have $0 < \left|\frac{x-1}{(\sqrt{2x+3}+\sqrt{5})}\right| < \frac{\varepsilon}{2}$

$$\therefore \lim_{x\to 1}\sqrt{2x+3} = \sqrt{5}$$

**倒推法总结：**

(1) 明确 $f(x), L, C, \varepsilon$

(2) 利用$|f(x) - L|$找出$|x - c|$
即找出$\delta$

(3) 经常将$|x - c|$的表达式写成乘法形式

$$\begin{cases} 二次函数: |x^2 - 1| \to |x+1||x-1| \\ 分式函数: \left|\dfrac{x-1}{x-2}\right| \to |x-1|\left|\dfrac{1}{x-2}\right| \\ 根式函数: \left|\dfrac{x-1}{(\sqrt{2x+3}+\sqrt{5})}\right| \to |x-1|\left|\dfrac{x-1}{(\sqrt{2x+3}+\sqrt{5})}\right| \end{cases}$$

**Example 11:** exponential function $\lim_{x\to 1}(10^x - 9) = 1$

*<u>Solution</u>* (a) $f(x) = 10^x - 9$, (b) $L = 1$, (c) $c = 1$ (d) ε, find $\delta$.
For each $\varepsilon > 0$, to prove that $|f(x) - L| = |(10^x - 9) - 1| < \varepsilon$

$$\xLeftarrow{if\ x>1} |10^x - 10| < \varepsilon$$

$$\Leftarrow 10^x - 10 < \varepsilon$$

$$\Leftarrow 10^x < 10 + \varepsilon$$

$$\Leftarrow x < \lg(10 + \varepsilon)$$

Note

$\Leftarrow x-1<\lg(10+\varepsilon)-1$

There exists $0>\delta=\lg\left(10+\frac{\varepsilon}{2}\right)-1<\lg(10+\varepsilon)-1$,when $0<|x-1|<\delta$, we have $x-1<\lg(10+\varepsilon)-1$

$\therefore \lim\limits_{x\to1}(10^x-9)=1$

$\xLeftarrow{if\ 0<x<1} |10^x-10|<\varepsilon$

$\Leftarrow 10-10^x<\varepsilon$
$\Leftarrow 10-\varepsilon<10^x$
$\Leftarrow \lg(10-\varepsilon)<x$
$\Leftarrow \lg(10-\varepsilon)-1<x-1$

There exists $\lg(10-\varepsilon)-1<\lg\left(10-\frac{\varepsilon}{2}\right)-1=\delta$ ,when $0<|x-1|<\delta$, we have $\lg(10-\varepsilon)-1<x-1$

$\therefore \lim\limits_{x\to1}(10^x-9)=1$

**Example 12:** trigonometric function $\lim\limits_{x\to\pi/4}\sin x=\frac{\sqrt{2}}{2}$

*<u>Solution</u>* (a) $f(x)=\sin x$, (b) $L=\frac{\sqrt{2}}{2}$, (c) $c={}^{\pi}/_{4}$ (d) ε, find $\delta$.

For each $\varepsilon>0$, to prove that $|f(x)-L|=\left|\sin x-\frac{\sqrt{2}}{2}\right|<\varepsilon$

$$\Leftarrow \left|\sin x-\sin\frac{\pi}{4}\right|<\varepsilon$$

$$\Leftarrow 2\left|\cos\frac{x+\frac{\pi}{4}}{2}\sin\frac{x-\frac{\pi}{4}}{2}\right|<\varepsilon$$

$$\Leftarrow \left|\cos\frac{x+\frac{\pi}{4}}{2}\right|\left|\sin\frac{x-\frac{\pi}{4}}{2}\right|<\frac{\varepsilon}{2}$$

$$\Leftarrow \begin{cases} when\ x\to\frac{\pi}{4}, \cos\frac{x+\frac{\pi}{4}}{2}<1 \\ \frac{\varepsilon}{2}=1\left(\frac{\varepsilon}{2}\right)=1\delta \end{cases}$$

There exists $0>\delta=\arcsin\frac{\varepsilon}{2}$ , when $0<\left|x-\frac{\pi}{4}\right|<\delta$, we have

$$\left|\cos\frac{x+\frac{\pi}{4}}{2}\right|\left|\sin\frac{x-\frac{\pi}{4}}{2}\right|<\frac{\varepsilon}{2}$$

$$\therefore \lim_{x\to\pi/4}\sin x=\frac{\sqrt{2}}{2}$$

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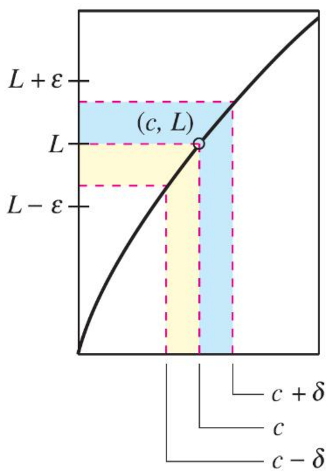



## Limit(极限)

Let *f(x)* be a function defined on an open interval containing *c* (except possibly at

Note

$c$) and let $L$ be a real number. The statement

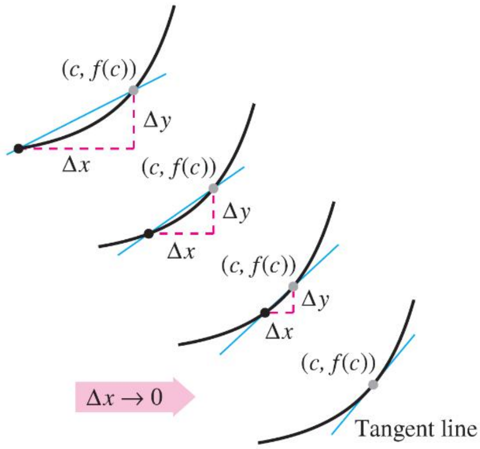
$$\lim_{x \to c} f(x) = L$$

means that for each ε > 0 there exists a δ > 0 such that

if $\quad 0 < |x - c| < \delta$

then $\quad |f(x) - L| < \varepsilon$

# The Derivative of a function(函数的导数)




1. The derivative of function *f(x)* at *x* is given by(x 点的导数)
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
2. For all *x* for which this limit exists, $f'(x)$ is a function of *x*.(导函数)
3. Notation for derivatives

$$f'(x),\ \frac{dy}{dx},\ y',$$

$$\frac{d}{dx}[f(x)],\ D_x[y]$$

# Differential of a function(函数的微分)

The differential is defined by

$$dy = f'(x)dx$$

The notion $dy/dx$ is read as the *derivative of y with respect to x* or *dy,dx*

# Antiderivatives and Indefinite Integration(反导数和不定积分)

A function *F(x)* is an **antiderivative(反导数)** of *f(x)* on interval *I* if

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } I.$$

The operation of finding all solutions of the differential equation

$$\frac{dy}{dx} = f(x) \qquad \text{or} \qquad dy = f(x)dx$$

is called **antidifferentiation** or **indefinite integration(不定积分)**. It is denoted by an integral sign $\int$

$$y = \int f(x)dx = F(x) + C$$

The expression $\int f(x)dx$ is read as the *antiderivative of f(x) respect to x*.

# Definite Integration(定积分)

Note

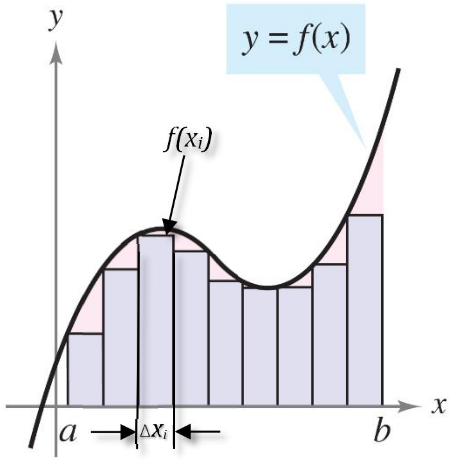
Sigma Notation: The sum of $n$ terms $a_1, a_2, a_3, \ldots a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

If $f(x)$ is defined on the closed interval $[a,b]$ and the limit of

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

is called the **definite integral** of $f(x)$ from $a$ to $b$.




# Infinite Series(无穷级数)

If $\{a_n\}$ is an infinite sequence $a_1, a_2, a_3, \ldots a_n \ldots$, then

$$\sum_{i=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is an **infinite series** (or simply a series).

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Note

**Exercise 1:** Given the limit $\lim_{x\to 2}\left(\frac{1}{x-1}\right) = 1$, find $\delta$ such that $\left|\frac{1}{x-1}\right| < 0.01$ whenever $0 < |x-2| < \delta$

<u>*Solution*</u>

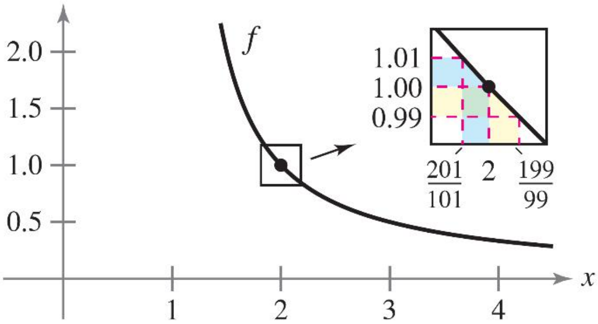
(a) $f(x) =$

(b) $L =$

(c) $\varepsilon =$

(d) c $=$

find $\delta$.




**Exercise 2:** Find the limit $L$. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x-c| < \delta$

(a) $\lim_{x\to 4}\left(4 - \frac{x}{2}\right)$

<u>*Solution*</u> (a) $f(x) =$ (b) $L =$ (c) c $=$ (d) ε find $\delta$.

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Note

(b) $\lim_{x \to 5} (x^2 + 4)$

*Solution* (a) $f(x) =$ (b) $L =$ (c) $c =$ (d) $\varepsilon$ find $\delta$.

**Exercise 3:** Find the limit $L$, then use the $\varepsilon - \delta$ definition to prove that the limit is $L$.

(a) $\lim_{x \to 0} \sqrt[3]{x}$

*Solution* (a) $f(x) =$ (b) $L =$ (c) $c =$ (d) $\varepsilon$ find $\delta$.

For each $\varepsilon > 0$, to prove that $|f(x) - L| =$ $< \varepsilon$

$\Leftarrow$

$\Leftarrow$

$\Leftarrow$

…

(b) $\lim_{x \to 4} \sqrt{x - 2}$

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Note

(c) $\lim_{x \to 2} x^2 - 1$

(d) $\lim_{x \to 4} (\frac{x-3}{x+1})$

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